A systematic approach to the evaluation of $\Sigma_{(m,n>0)}(am^2+bmn+cn^2)^{-s}$

1976 J. Phys. A: Math. Gen. 9 1215

(http://iopscience.iop.org/0305-4470/9/8/007)

J. Phys. A: Math. Gen., Vol. 9, No. 8, 1976. Printed in Great Britain. © 1976

A systematic approach to the evaluation of $\sum_{(m,n\neq 0,0)} (am^2+bmn+cn^2)^{-s}$

I J Zucker† and M M Robertson‡

† Department of Physics, University of Surrey, Guildford GU2 5XH, UK
‡ Department of Mathematics, University of Surrey, Guildford GU2 5XH, UK

Received 23 December 1975, in final form 6 April 1976

Abstract. A condition proposed by Glasser for the double sum

$$S=\sum_{(m,n\neq 0,0)}(am^2+bmn+cn^2)^{-s}$$

to be decomposable into sums of products of simple sums is examined. The condition is that the reduced forms of the binary quadratic form $am^2+bmn+cn^2$ be divisible so that there is just one class per genus. Although no general proof has yet been provided, procedures are described here for decomposing and evaluating any S satisfying the suggested condition, thus demonstrating that the condition is sufficient. Experience with double sums not satisfying the condition suggests that it is also necessary.

1. Introduction

In a previous paper Zucker and Robertson (1975, to be referred to as I) evaluated exactly a few double sums of general form

$$S=S(a,b,c)=S(a,b,c:s)=\sum_{(m,n,\neq 0,0)}(am^2+bmn+cn^2)^{-s} \tag{1.1}$$

$$S_1=S_1(a,b,c)=S_1(a,b,c:s)=\sum_{(m,n\neq 0,0)}(-1)^m(am^2+bmn+cn^2)^{-s} \tag{1.2}$$

$$S_2=S_2(a,b,c)=S_2(a,b,c:s)=\sum_{(m,n\neq 0,0)}(-1)^n(am^2+bmn+cn^2)^{-s} \tag{1.3}$$

$$S_{1,2}=S_{1,2}(a,b,c)=S_{1,2}(a,b,c:s)=\sum_{(m,n\neq 0,0)}(-1)^{m+n}(am^2+bmn+cn^2)^{-s}. \tag{1.4}$$

The summation is over all integer values of m and n excluding the case where both are simultaneously zero. The term 'exact' is used here in the sense introduced by Glasser (1973b), and means that S may be decomposed into a linear sum of products of pairs of Dirichlet L-series. If this can be done S is said to be solvable. The properties of L-series have been discussed in the previous communication (Zucker and Robertson 1976, to be referred to as II). In I two queries were raised. These were: (a) no criterion was known for when any given S was soluble; (b) no general methods for solving S could be given. We believe it is now possible to answer both these queries. To do this properties of the binary quadratic form $am^2+bmn+cn^2=(a,b,c)$ associated with S have to be discussed and this is now done.

The form (a, b, c) in which a, b, c, m, n are all integers has a *discriminant*, $d = b^2 - 4ac = -\Delta$. If $d < 0$ $(\Delta > 0)$ the form is said to be *definite*. If $a > 0$ the form is positive definite. Only positive definite forms will be considered here. If $b \leqslant a \leqslant c$ the form is said to be *reduced*. For any given d there is only a finite number of reduced forms. This is called the *class number*, $h(d)$, of the discriminant d. If a, b, c are relatively prime the form is said to be *primitive*, otherwise it is *imprimitive*. We write $H(d)$ for the number of primitive reduced forms and $I(d)$ for the number of imprimitive reduced forms. As an example consider the forms with $d = -12$. There are two reduced forms, $(1, 0, 3)$ and $(2, 2, 2)$, hence $h(-12) = 2$. $(1, 0, 3)$ is pimitive and $(2, 2, 2)$ is imprimitive so that in this case $H(-12) = I(-12) = 1$. All other (a, b, c) with $d = -12$ can be reduced to either $(1, 0, 3)$ or $(2, 2, 2)$ by unimodular transformations. For example, $(1, 4, 7)$ will be reduced to $(1, 0, 3)$ by the substitution $m = M - 2N$, $n = N$. Such forms are said to be equivalent, and are written $(1, 0, 3) \sim (1, 4, 7)$. The form (a, b, c) is said to *represent* the number X if there exist integers m, n such that $am^2 + bmn + cn^2 = X$. Equivalent forms represent the same integers. The value of $S(a, b, c)$ is thus the same for all equivalent forms and therefore we need only concern ourselves with reduced (a, b, c). The reduced forms are divided into *genera* according to the integers represented. Every genus has an equal number of reduced forms. If for some discriminant there is precisely one form per genus the forms of that discriminant are said to be *disjoint*. For positive definite forms Δ can only be either $\equiv 0 \pmod 4$ or $\equiv 3 \pmod 4$. According to whether $\Delta \equiv 0 \pmod 4$ or $3 \pmod 4$, $(1, 0, \Delta/4)$ or $(1, 1, \frac{1}{4}(\Delta + 1))$ is called the *principal* form of discriminant Δ. A considerable amount of the theory of binary quadratic forms has been summarized very briefly in the foregoing account. A fuller account may be obtained from Dickson (1929, 1939).

Smart (1973) and Glasser (1973a, b) have revived much recent interest in the evaluation of S. In particular Glasser (1973b) solved $S(1, 0, 16)$ using number theoretic techniques. The discriminant of $(1, 0, 16)$ is -64, and the only other reduced form of that discriminant is $(4, 4, 5)$. $(1, 0, 16)$ and $(4, 4, 5)$ are disjoint and Glasser (1973b) suggested that S might always be solvable whenever the reduced forms associated with it were disjoint. We have in fact been able to solve S in every case given in Dickson (1929) where this is so. Over a hundred cases have been given there. As yet it has not been possible to solve any other S in terms of L-series with real characters. Thus, it appears to us that Glasser's (1973b) criterion for the solvability of S is correct. However, we know of no complete proof of this. The main purpose of this communication is to present general methods of solving S when this is possible, and so answer the second query raised in I. The solution for the principal reduced form for a given Δ will be mainly considered, but there is no difficulty in extending the methods used here to the other reduced forms.

2. Solutions for the principal form of $S(a, b, c)$

All the solutions have been exhibited in tables 1 and 2. It was found that most of the solvable S were divided into several species and sub-species each characterized by their own properties. The symbols used in the tables have the following meaning. $P = 1$ or $\Pi'_{i=1} p_i$ where the p_i are all *different odd* primes, i.e. P is an odd square-free integer. The summation $\Sigma_{\mu|P}$ is over all the divisors, μ, of P. 1 is included amongst the divisors. $(2|\mu)$ is the Kronecker symbol as defined in II. α is written for $[(2|\mu) + (2|P/\mu)]$ and can assume one of the three values $0, \pm 2$. β is written for $(2|\mu)\,(2|P/\mu)$ and may be ± 1. The

Table 1. Characteristics of species.

Species	Δ	Restriction on P	Principal form	H	I	Principal solution
1	P	$P\equiv 3 \pmod 4$	$[1, 1, (1+P)/4]$	2^{n-1}	0	$2^{n-2}S(1,1,(1+P)/4)=\sum_{\mu\mid P} L_{\pm\mu}L_{\mp P/\mu}$
2	$4P$	$P\equiv 1 \pmod 4$	$(1, 0, P)$	2^n	0	$2^{n-1}S(1,0,P)=\sum_{\mu\mid P} L_{\pm\mu}L_{\mp 4P/\mu}$
3(a)	$8P$		$(1, 0, 2P)$	2^n	0	$2^{n-1}S(1,0,2P)=\sum_{\mu\mid P} L_{\pm\mu}L_{\mp 8P/\mu}$
3(b)	$32P$		$(1, 0, 8P)$	2^{n+1}	2^n	$2^nS(1,0,8P)=\sum_{\mu\mid P}\{[1-(2\mid\mu)2^{-s}+2^{1-2s}]L_{\pm\mu}L_{\mp 8P/\mu}+L_{\mp 4\mu}L_{\pm 8P/\mu}\}$
4(a)	$4P$	$P\equiv 3 \pmod 4$	$(1, 0, P)$	2^{n-1}	2^{n-1}	$2^{n-2}S(1,0,P)=\sum_{\mu\mid P}(1-\alpha 2^{-s}+2^{1-2s})L_{\pm\mu}L_{\mp p/\mu}$
4(b)	$16P$	$P\equiv 3 \pmod 4$	$(1, 0, 4P)$	2^n	2^n	$2^{n-1}S(1,0,4P)=\sum_{\mu\mid P}\{[1-\alpha 2^{-s}+(2+\beta)2^{-2s}-\alpha 2^{1-3s}+2^{2-4s}]$ $\times L_{\pm\mu}L_{\mp P/\mu}+L_{\mp 4}L_{\pm 4P/\mu}\}$
4(c)	$64P$	$P\equiv 3 \pmod 4$	$(1, 0, 16P)$	2^{n+1}	2^{n+1}	$2^nS(1,0,16P)=\sum_{\mu\mid P}\{[1-\alpha 2^{-s}+(2+\beta)2^{-2s}-\alpha 2^{1-3s}+(2+\beta)^{1-4s}$ $-\alpha 2^{2-5s}+2^{3-6s}]L_{\pm\mu}L_{\mp p/\mu}+(1+2^{1-2s})L_{\mp 4\mu}L_{\pm 4P/\mu}$ $+L_{+8\mu}L_{-8P/\mu}+L_{-8\mu}L_{+8P/\mu}\}$

Table 2.

(a) Solutions for special $S(1,0,\lambda)$

λ	$S(1,0,\lambda)$	λ	$S(1,0,\lambda)$
1	$4L_{+1}L_{-4}$	25	$(1-2\,.\,5^{-s}+5^{1-2s})L_{+1}L_{-4}+L_{+5}L_{-20}$
4	$2(1-2^{-s}+2^{1-2s})L_{+1}L_{-4}$	45	$2^{-1}[(1-2\,.\,3^{-s}+3^{1-2s})L_{+1}L_{-20}+L_{-3}L_{+60}$
9	$(1+3^{1-2s})L_{+1}L_{-4}+L_{-3}L_{+12}$		$+(1+2\,.\,3^{-s}+3^{1-2s})L_{-4}L_{+5}+L_{+12}L_{-15}]$
16	$(1-2^{-s}+2^{1-2s}-2^{1-3s}+2^{2-4s})L_{+1}L_{-4}$	72	$2^{-1}[(1-2\,.\,3^{-s}+3^{1-2s})(1-2^{-s}+2^{1-2s})L_{+1}L_{-8}$
	$+L_{+8}L_{-8}$		$+(1+2\,.\,3^{-s}+3^{1-2s})L_{-4}L_{+8}$
18	$(1-2\,.\,3^{-s}+3^{1-2s})L_{+1}L_{-8}+L_{-3}L_{+24}$		$+(1+2^{-s}+2^{1-2s})L_{-3}L_{+24}+L_{+12}L_{-24}]$

(b) Solutions for special $S(1,1,\lambda)$

λ	Δ	
7	27	$2(1-2\,.\,3^{-s}+3^{1-2s})L_{+1}L_{-3}$
19	75	$(1+5^{1-2s})L_{+1}L_{-3}+L_{+5}L_{-15}$
25	99	$(1-2\,.\,3^{-s}+3^{1-2s})L_{+1}L_{-11}+L_{-3}L_{+33}$
37	147	$(1-2\,.\,7^{-s}+7^{1-2s})L_{+1}L_{-3}+L_{-7}L_{+21}$
79	315	$2^{-1}[(1-2\,.\,3^{-s}+3^{1-2s})L_{+1}L_{-35}+(1+2\,.\,3^{-s}+3^{1-2s})L_{+5}L_{-7}+L_{-3}L_{+105}+L_{-15}L_{+21}]$

product of two L-functions is written $L_{\pm\mu}L_{\mp\nu}$ in order to emphasize the fact that the pairs of L-function appearing in the solution are *always* of opposite type as defined in II. Examples of solutions follow.

Species 1

$$P=7=\Delta,\ n=1: \qquad S(1,1,2)=2L_{+1}L_{-7}$$

$$P=3\times5,\ n=2: \qquad S(1,1,4)=L_{+1}L_{-15}+L_{-3}L_{+5}.$$

Dickson (1929) has given 31 examples of this species with $P<23\,000$. They are $P=3$, 7, 11, 15, 19, 35, 43, 51, 67, 91, 115, 123, 163, 187, 195, 235, 267, 403, 427, 435, 483, 555, 595, 627, 715, 795, 1155, 1435, 1995, 3003, 3315. The solution for $P=3$ is a special case. It is $S(1,1,1)=6L_{+1}L_{-3}$.

Species 2

$$P=5,\ \Delta=20,\ n=1: \qquad S(1,0,5)=L_{+1}L_{-20}+L_{-4}L_{+5}.$$

Dickson (1929) gives 18 numbers of this species with $P<10\,000$. They are $P=5$, 13, 21, 33, 37, 57, 85, 93, 105, 133, 165, 177, 253, 273, 345, 357, 385, 1365.

Species 3(a)

$$P=3,\ \Delta=24,\ n=1: \qquad S(1,0,6)=L_{+1}L_{-24}+L_{-3}L_{+8}$$

Species 3(b)

$$P=3,\ \Delta=96,\ n=1: \qquad (2|1)=+1,\ (2|3)=-1$$

$$2S(1,0,24)=(1-2^{-s}+2^{1-2s})L_{+1}L_{-24}+(1+2^{-s}+2^{1-2s})L_{-3}L_{+8}+L_{-4}L_{+24}+L_{+12}L_{-8}.$$

There are 15 examples of this species with $P<10\,000$. They are $P=1$, 3, 5, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231.

Species 4(*a*)

$$P=7,\ \Delta=28,\ n=1: \qquad (2|1)=1,\ (2|7)=1,\ \alpha=2,\ \beta=1$$

$$S(1,0,7)=2(1-2^{1-s}+2^{1-2s})L_{+1}L_{-7}$$

Species 4(*b*)

$$\Delta=112$$

$$S(1,0,28)=(1-2^{1-s}+3\,.\,2^{-2s}-2^{2-3s}+2^{2-4s})L_{+1}L_{-7}+L_{-4}L_{+28}$$

Species 4(*c*)

$$\Delta=448$$

$$2S(1,0,112)=(1-2^{1-s}+3\,.\,2^{-2s}-2^{2-3s}+3\,.\,2^{1-4s}-2^{3-5s}+2^{3-6s})L_{+1}L_{-7}$$
$$+(1+2^{1-2s})L_{-4}L_{+28}+L_{-8}L_{+56}+L_{+8}L_{-56}.$$

There are only three members of this species for $P<10\,000$ and they are $P=3,7,15$.

The method for solving S for species 1, 2 and 3(*a*) for which all the reduced forms are primitive is essentially the arithmetic method described by Glasser (1973b). The method is now illustrated here with proofs for species 1 when first $P=p$ and then $P=pq$, p and q being different odd primes.

For $P=p$ there is just one reduced form, f and this is $f=(1,1,\tfrac{1}{4}(1+P))$. Let $f(N)$ be the number of ways the integer N can be represented by f. Suppose $m^2+mn+(1+p)n^2/4\equiv 0 \pmod p$, then

$$m^2+mn(p+1)+(p+1)^2n^2/4\equiv 0 \pmod p$$

since we have just added $pmn+(p^2+p)n^2/4$. Therefore

$$[m+(p+1)n/2]^2\equiv 0 \pmod p \qquad \text{and} \qquad m+(p+1)n/2\equiv 0 \pmod p.$$

It follows that $m-(p-1)n/2\equiv 0 \pmod p$. Let

$$m+(p+1)n/2=px \qquad \text{and} \qquad m-(p-1)n/2=py.$$

It then follows that

$$m^2+mn+(p+1)n^2/4=(p^2+p)(x^2+y^2)/4+(p^2-p)xy/2.$$

Now let $x=u$ and $y=-u-v$, then

$$m^2+mn+(p+1)n^2/4=p[u^2+uv+(p+1)v^2/4]. \tag{2.1}$$

(2.1) shows that to every (m,n) such that $m^2+mn+(1+p)n^2/4=pN$ there is a unique (u,v) such that $u^2+uv+(p+1)v^2/4=N$ and vice versa. This means that $f(N)=f(pN)$. Therefore by theorem 64 in Dickson (1929), which is a modified form of a theorem first given by Dirichlet (1840),

$$S(1,1,\tfrac{1}{4}(1+p))=\frac{2}{1-p^{-s}}\sum_{(n,p)=1}n^{-s}\sum_{\mu|n}(-p|n)$$
$$=\frac{2}{1-p^{-s}}\sum_{(\mu,p)}(\mu|p)\mu^{-s}\sum_{(r,p)=1}r^{-s}=2L_{+1}L_{-p}. \tag{2.2}$$

When $P=pq$ the proof is more complicated. There are two reduced forms namely $f=(1,1,\tfrac{1}{4}(pq+1))$ and $g=(\tfrac{1}{4}(p+q),\tfrac{1}{2}(p-q),\tfrac{1}{4}(p+q))$. It is first proved as before that

$f(N)=g(pN)=g(qN)$ and that $g(N)=f(pN)=f(qN)$. Hence $f(N)=f(p^2N)=f(q^2N)=f(pqN)$, with similar relations for $g(N)$. Then by the Dirichlet theorem

$$S(1,1,\tfrac{1}{4}(pq+1))=(1-p^{-2s})^{-1}(1-q^{-2s})^{-1}\sum_{(n,pq)=1} n^{-s}$$
$$\times\{[1+(pq)^{-s}][1+(n|P)]+[p^{-s}+q^{-s}][1-(n|P)]\}\sum_{\mu|n}(-pq|\mu)$$
$$=A+B \tag{2.3}$$

$$A=(1-p^{-s})^{-1}(1-q^{-s})^{-1}\sum_{(\mu,pq)=1}\mu^{-s}(-pq|\mu)\sum_{(r,pq)=1}r^{-s}=L_{+1}L_{-pq} \tag{2.4}$$

$$B=(1+p^{-s})^{-1}(1+q^{-s})^{-1}\sum_{(\mu,pq)=1}\mu^{-s}(-pq|\mu)\sum_{(r,pq)=1}r^{-s}(r|p)$$
$$=\sum_{(\mu,q)=1}\mu^{-s}(\mp q|\mu)\sum_{(r,p)=1}r^{-s}(\pm p|r)=L_{\mp q}L_{\pm p}. \tag{2.5}$$

The non-principal solution is

$$S(\tfrac{1}{4}(p+q),\tfrac{1}{2}(p-q),\tfrac{1}{4}(p+q))=A-B=L_{+1}L_{-pq}-L_{\mp q}L_{\pm p}. \tag{2.6}$$

Thus in the example given for species 1 in which $P=3\times 5$ the non-principal solution is

$$S(2,1,2)=L_{+1}L_{-15}-L_{-3}L_{+5}. \tag{2.7}$$

The solutions for species 2 and 3(a) follow a similar procedure. The solution becomes progressively more complicated as the number of primes increase, but nonetheless conforming to the pattern illustrated above. There is no difficulty in obtaining the solutions for non-principal forms.

Species 3(b), 4(a), 4(b) and 4(c) have imprimitive reduced forms associated with them. This makes the solution of such S by the arithmetic method just described very complex. The solution for some members of species 4(a) was carried out by this approach. However, the solutions for species 3(b), 4(b) and 4(c) were obtained in a completely different fashion. This used the Jacobian θ-functions and this method of evaluating multiple sums has been discussed by Glasser (1973a) and Zucker (1974). The approach is as follows.

θ-functions of zero argument may be expressed as infinite series of a parameter q. They are as given by Whittaker and Watson (1958)

$$\theta_2=\theta_2(q)=\sum_{-\infty}^{\infty}q^{(n-\frac{1}{2})^2};\qquad \theta_3=\theta_3(q)=\sum_{-\infty}^{\infty}q^{n^2};\qquad \theta_4=\theta_4(q)=\sum_{-\infty}^{\infty}(-1)^n q^{n^2}. \tag{2.8}$$

Let the Mellin operator, M_s, be defined as

$$\Gamma(s)\mathrm{M}_s(f(t))=\int_0^{\infty}t^{s-1}f(t)\,\mathrm{d}t. \tag{2.9}$$

It then follows that

$$S(1,0,\lambda)=\mathrm{M}_s(\theta_3\theta_3(q^{\lambda})-1) \tag{2.10}$$
$$S_1(1,0,\lambda)=\mathrm{M}_s(\theta_4\theta_3(q^{\lambda})-1) \tag{2.11}$$
$$S_2(1,0,\lambda)=\mathrm{M}_s(\theta_3\theta_4(q^{\lambda})-1) \tag{2.12}$$
$$S_{1,2}(1,0,\lambda)=\mathrm{M}_s(\theta_4\theta_4(q^{\lambda})-1)\qquad \text{where } q=\mathrm{e}^{-t}. \tag{2.13}$$

If a series for $\theta_3\theta_3(q^\lambda)$ can be obtained then $S(1,0,\lambda)$ may be obtained immediately. Indeed $S(1,0,3)$ was first solved using a q-series for $\theta_3\theta_3(q^3)$ originally given by Cauchy (1844). The converse is true. That is, knowing the solution to $S(1,0,\lambda)$ it is simple to establish the q-series for $\theta_3\theta_3(q^\lambda)$. In certain cases it is then possible to derive q-series for $\theta_3\theta_3(q^{4\lambda})$ and $\theta_3\theta_3(q^{16\lambda})$ from the known series for $\theta_3\theta_3(q^\lambda)$ and so solve $S(1,0,4\lambda)$ and $S(1,0,16\lambda)$. This 'four-folding' method proceeds as follows. Three well known identities for θ-function are employed. They are:

$$\begin{aligned} &(a) \qquad \theta_3=\theta_3(q^4)+\theta_2(q^4);\\ &(b) \qquad 2\theta_3(q^4)=\theta_3+\theta_3;\\ &(c) \qquad \theta_3(-q)=\theta_4(q). \end{aligned} \tag{2.14}$$

τq is substituted for q in the known series for $\theta_3\theta_3(q^\lambda)$, where $\tau^4=-1$. Then using (2.14a) we obtain

$$\theta_3(\tau q)\theta_3((\tau q)^\lambda)=(\theta_4(q^4)+\tau\theta_2(q^4))\{\theta_3[(-1)^\lambda q^{4\lambda}]+\tau^\lambda\theta_2(q^{4\lambda})\}. \tag{2.15}$$

Let $\lambda\equiv 2 \pmod 4$, then the coefficient of τ, $C_\tau(q)$, in (2.15) is

$$C_\tau(q)=\theta_2(q^4)\theta_3(q^{4\lambda})=[\theta_3-\theta_3(q^4)]\theta_3(q^{4\lambda}).$$

Hence

$$\begin{aligned} \mathrm{M}_s(C_\tau(q))&=\mathrm{M}_s(\theta_3\theta_3(q^{4\lambda})-\theta_3(q^4)\theta_3(q^{4\lambda}))\\ &=S(1,0,4\lambda)-2^{-2s}S(1,0,\lambda). \end{aligned} \tag{2.16}$$

If $\lambda=3 \pmod 4$ then

$$C_\tau(q)=\theta_2(q^4)\theta_4(q^{4\lambda}). \tag{2.17}$$

τq is again substituted for q in (2.17) and the new coefficient of τ, $D_\tau(q)$, found. Then

$$D_\tau(q)=\theta_2(q^4)\theta_3(q^{4\lambda}); \qquad \mathrm{M}_s(D_\tau(q))=S(1,0,4\lambda)-2^{-2s}S(1,0,\lambda). \tag{2.18}$$

Further for $\lambda\equiv 3 \pmod 4$

$$C_\tau(q)+D_\tau(q)=\theta_2(q^4)(\theta_3(q^{4\lambda})+\theta_4(q^{4\lambda}))=2\theta_2(q^4)\theta_3(q^{16\lambda})$$

and

$$\mathrm{M}_s(C_\tau(q)+D_\tau(q))=2(S(1,0,16\lambda)-2^{-2s}S(1,0,4\lambda)). \tag{2.19}$$

It is simple to determine C_τ and D_τ in the above cases. Hence in the case $\lambda\equiv 2 \pmod 4$ the derivation of $S(1,0,4\lambda)$ from $S(1,0,\lambda)$ is equivalent to solving species 3(b) knowing the solution to 3(a). Similarly for $\lambda\equiv 3 \pmod 4$ (2.18) and (2.19) provide the solutions for species 4(b) and 4(c) from the known solution of 4(a). Table 3 shows how products of pairs of L-functions transform after substituting τq for q in their q-series representations.

Table 3. Transformation of products of L-series; p, q are odd.

L-function product	$2\mathrm{M}_s(C_\tau(q))$	$2\mathrm{M}_s(D_\tau(q))$
$L_{\pm p}L_{\mp q}$	$L_{+8p}L_{-8q}+L_{-8p}L_{+8q}$	$[(1-2\|p)2^{-s}][1-(2\|q)2^{-s}]L_{\pm p}L_{\mp q}+L_{\mp 4p}L_{\pm 4q}$
$L_{\pm p}L_{\mp 8q}$	$[1-(2\|q)2^{-s}]L_{\pm p}L_{\mp 8q}+L_{\mp 8p}L_{\pm 4q}$	

3. Miscellaneous solvable S

Several S with disjoint forms remain which do not belong to any of the species discussed. They divide into two groups. Group (*a*) has $\Delta = 4$, 16, 36, 64, 72, 100, 180 and 288 with principal form $(1, 0, \Delta/4)$ where $\Delta/4$ is *not* square free. Group (*b*) has $\Delta = 27$, 75, 99, 147 and 315 with principal form $(1, 1, \frac{1}{4}(\Delta+1))$ where Δ is *not* square free. All have been solved by various methods. $S(1, 0, 1)$ is the best known of all these results and its solution was given first by Hardy (1919). Glasser (1973b) solved $S(1, 0, 4)$ and $S(1, 0, 16)$. Both may be obtained from $S(1, 0, 1)$ by the 'four-folding' method previously described, as this method works in the particular case $\lambda = 1$. A 'nine-folding' method similar to 'four-folding' can be deduced. This makes use of the substitution ωq for q in a known series for $\theta_3\theta_3(q^\lambda)$, where $\omega^3 = 1$. If $\lambda \equiv 2 \pmod 3$ finding, $C_\omega(q)$, the coefficient of ω in the q-series for $\theta_3\theta_3(q^\lambda)$, leads to

$$\mathrm{M}_s[C_\omega(q)] = S(1, 0, 9\lambda) - 3^{-2s} S(1, 0, \lambda). \tag{3.1}$$

$S(1, 0, 2)$, $S(1, 0, 5)$ and $S(1, 0, 8)$ have been solved and hence the solutions to $S(1, 0, 18)$, $S(1, 0, 45)$ and $S(1, 0, 72)$ are obtained. $S(1, 0, 25)$ and all the members of group (*b*) were solved using the arithmetic method. All these results have been given in table 2.

4. The sums S_1, S_2 and $S_{1,2}$

No systematic approach to the solutions of S_1, S_2 and $S_{1,2}$ has been attempted. Relations amongst them have been found, and a few algorithms developed. These are described below.

For sums with principal form $(1, 0, \lambda)$ if we add together (2.10), (2.11), (2.12) and (2.13) we obtain the relation

$$S(1, 0, \lambda) + S_1(1, 0, \lambda) + S_2(1, 0, \lambda) + S_{1,2}(1, 0, \lambda) = 2^{2-2s} S(1, 0, \lambda). \tag{4.1}$$

Hence only three of the four sums are independent. Similarly adding (2.10) to (2.12) and (2.11) to (2.13) yields the relations

$$S(1, 0, \lambda) + S_2(1, 0, \lambda) = 2S(1, 0, 4\lambda) \tag{4.2}$$

$$S_1(1, 0, \lambda) + S_{1,2}(1, 0, \lambda) = 2S_1(1, 0, 4\lambda). \tag{4.3}$$

Put $-q$ for q in equations (2.10) to (2.13) and we obtain for

$$\begin{array}{lll} \lambda \text{ even} & S(1, 0, \lambda) \to S_1(1, 0, \lambda); & S_2(1, 0, \lambda) \to S_{1,2}(1, 0, \lambda) \\ \lambda \text{ odd} & S(1, 0, \lambda) \to S_{1,2}(1, 0, \lambda); & S_1(1, 0, \lambda) \to S_2(1, 0, \lambda). \end{array} \tag{4.4}$$

The effect of putting $-q$ for q into the q-series representation of an L-function is to transform $L_{\pm k}$ if k is odd according to the rule

$$L_{\pm k} \to -[1 - (2|k) 2^{1-s}] L_{\pm k}. \tag{4.5}$$

If k is even, or the function is multiplied by any factor involving 2^{-s} it is unchanged. As an example consider $S(1, 0, 5) = L_{+1}L_{-20} + L_{-4}L_{+5}$. By putting $-q$ for q into the q-series for this sum we obtain by (4.4) $S_{1,2}(1, 0, 5)$. Thus (4.5) allows us to write down its value right away, namely

$$S_{1,2}(1, 0, 5) = -[(1 - 2^{1-s}) L_{+1}L_{-20} + (1 + 2^{1-s}) L_{-4}L_{+5}]. \tag{4.6}$$

Some relations between $S(1,0,\lambda)$ and $S(1,1,\lambda')$, where $\lambda'=(1+\lambda)/4$, may be obtained geometrically. Consider two-dimensional space divided into a rectangular lattice of sides 1 and $\sqrt{\lambda}$ (figure 1). Take any lattice point to be the origin, O. Then any other lattice point is a distance $(m^2+\lambda n^2)^{1/2}$ from O, where m and n can take all integer values, excluding the case where both are simultaneously zero. Let the lattice now have particles of the same sign placed on every lattice point and suppose the particles interact with a potential, r^{-2s}, r being the interparticle separation. Then the interaction of the particle at O with all other particles is just $S(1,0,\lambda)$. Form a new lattice from the original by putting lattice points on the diagonal bisectors of the old lattice (figure 2).

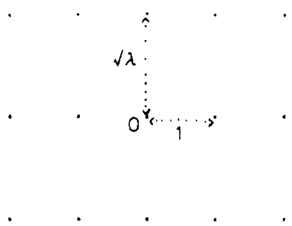

Figure 1. Geometric representation of $S(1,0,\lambda)$.

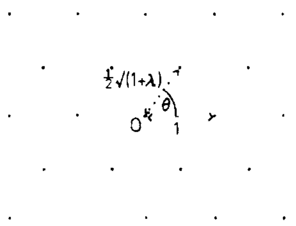

Figure 2. Geometric representation of $S[(1,1,(1+\lambda)/4]$.

The new lattice points are a distance $[(m-\frac{1}{2})^2+\lambda(n-\frac{1}{2})^2]^{1/2}$ from O. But *every* point in the new lattice is a distance $(m^2+mn+\lambda' n^2)^{1/2}$ from O. Hence if we put particles as before on the new lattice sites as well, the interaction of the origin particle with all the others is just $S(1,1,\lambda')$. But this is just the original interaction $S(1,0,\lambda)$ plus the interaction of the origin particle with those in the new lattice sites, i.e.

$$S(1,1,\lambda')=S(1,0,\lambda)+\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}[(m-\tfrac{1}{2})^2+\lambda(n-\tfrac{1}{2})^2]^{-s}. \tag{4.7}$$

The right-hand side (RHS) of (4.7) can be expressed as Mellin transforms of θ-functions giving

$$\text{RHS}(4.7)=\mathrm{M}_s(\theta_3\theta_3(q^\lambda)-1+\theta_2\theta_2(q^\lambda)). \tag{4.8}$$

Using the identities (2.14) enables us after some manipulation to write

$$\theta_3\theta_3(q^\lambda)+\theta_2\theta_2(q^\lambda)=\tfrac{1}{2}(\theta_3(q^{1/4})\theta_3(q^{\lambda/4})+\theta_4(q^{1/4})\theta_4(q^{\lambda/4})). \tag{4.9}$$

Then using the property of Mellin transforms that

$$\mathrm{M}_s(f(q^k))=k^{-s}\mathrm{M}_s(f(q)), \qquad q=\mathrm{e}^{-t} \tag{4.10}$$

we have

$$S(1,1,\lambda')=2^{2s-1}(S(1,0,\lambda)+S_{1,2}(1,0,\lambda)). \tag{4.11}$$

If instead of putting particles of the same sign on the new lattice sites we put particles of sign opposite to those on the original lattice sites, then the interaction of the origin particle with all others is $S_2(1, 1, \lambda')$. Then it may be shown in a similar fashion to before that

$$S_2(1, 1, \lambda') = 2^{2s-1}(S_1(1, 0, \lambda) + S_2(1, 0, \lambda)). \tag{4.12}$$

Adding (4.11) to (4.12) and using (4.1) we have

$$S(1, 1, \lambda') + S_2(1, 1, \lambda') = 2S(1, 0, \lambda). \tag{4.13}$$

Consider again the original rectangular lattice. Suppose we place particles of alternating sign on the lattice sites. The interaction of the origin particle with all others will then be $S_{1,2}(1, 0, \lambda)$. Let us now place particles of alternating sign on the diagonal bisectors. The array formed will appear as either figure 3(a) or figure 3(b).

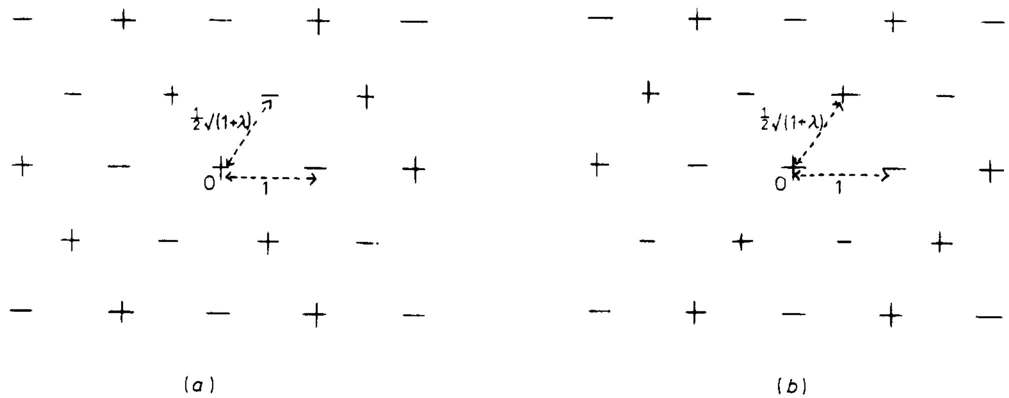

Figure 3. (a) Geometric representation of $S_{1,2}[(1, 1, (1+\lambda)/4]$; ($b$) Geometric representation of $S_1[1, 1, (1+\lambda)/4]$.

The interaction of the origin particle with all others is either $S_{1,2}(1, 1, \lambda')$ or $S_1(1, 1, \lambda')$. But the addition of particles with alternating signs on the diagonal bisectors contributes, from symmetry considerations, precisely *zero* to the original sum, i.e.

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(-1)^{m+n}[(m-\tfrac{1}{2})^2 + \lambda(n-\tfrac{1}{2})^2]^{-s} \equiv 0.$$

Hence

$$S_{1,2}(1, 1, \lambda') = S_1(1, 1, \lambda') = S_{1,2}(1, 0, \lambda). \tag{4.14}$$

If (4.13) is now added to twice (4.14) then

$$\begin{aligned} S(1, 1, \lambda') &+ S_1(1, 1, \lambda') + S_2(1, 1, \lambda') + S_{1,2}(1, 1, \lambda') \\ &= 2(S(1, 0, \lambda) + S_{1,2}(1, 0, \lambda)) = 2^{2-2s}S(1, 1, \lambda') \end{aligned} \tag{4.15}$$

which is analogous to (4.1). From these relations it is seen that of the four sums (1.1)–(1.4) only two are independent when $a = b = 1$, since (4.15) shows only three are independent, of which two are equal by (4.14). No doubt similar relations for sums related to non-principal forms may be found.

Numerically the evaluation of S requires the calculation of the L-series. Since all these series (except L_{+1}) consist of equal numbers of positive and negative terms periodically recurring, their evaluation is straightforward. For s an odd positive integer the a-type functions, and for s an even positive integer the b-type functions there are known formulae (II(4.11), (4.12)) giving the functions in terms of π. Since the pairs of L-series occurring in the solution of S are always of opposite type, no formula in terms of known simple transcendental numbers can be found for S with integral s. However, $s=1$ is a special case for both types of L-series are expressible in closed form (II(4.14), (4.15)). Hence for $s=1$, though S diverges, S_1, S_2 and $S_{1,2}$ may all be expressed in closed form. This leads to some unusual formulae, e.g.

$$\sum_{(m,n\neq 0,0)}(-1)^m(m^2+58n^2)^{-1}=\frac{-\pi}{\surd 58}\ln(27+5\surd 29). \tag{4.16}$$

Some interesting summation formulae and also approximations to π involving logarithms of surds similar to those obtained by Ramanujan (1914) may also be deduced. These will be discussed elsewhere.

5. Conclusion

The solution to some problems raised in I concerning the evaluation of S have been proposed. As yet there appears to be no practical significance (except elegance) in the results obtained. However, the fact that quite complex two-dimensional lattice sums may be decomposed into simple sums lends encouragement to the pursuit of similar solutions of the physically more important, but so far elusive, three-dimensional sums. A different way of constructing q-series without using θ-functions has recently been proposed by Glasser (1975), who thus solved a five-divisional sum. His cautious optimism concerning the solution of three- and other odd-dimensional results balances our pessimism—now slightly lessened—with regard to them.

Acknowledgments

We are most grateful to Professor M L Glasser for his interest and his continuing encouraging correspondence.

References

Cauchy A L 1844 *C. R. Acad. Sci., Paris* **19** 1385
Dickson L E 1929 *Introduction to the Theory of Numbers* (New York: Dover)
—— 1939 *Modern Elementary Theory of Numbers* (Chicago: University of Chicago Press)
Dirichlet L E 1840 *J. Math.* **21** 1–12
Glasser M L 1973a *J. Math. Phys.* **14** 409–13
—— 1973b *J. Math. Phys.* **14** 701–3
—— 1975 *J. Math. Phys.* **16** 1237–8
Hardy G H 1919 *Mess. Math.* **49** 85–91
Ramanujan S 1914 *Q. J. Math.* **45** 350–72
Smart J R 1973 *Glasg. Math. J.* **14** 1–12
Whittaker E T and Watson G N 1958 *Modern Analysis* (Cambridge: Cambridge University Press) chap 21
Zucker I J 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1568–75
Zucker I J and Robertson M M 1975 *J. Phys. A: Math. Gen.* **8** 874–81
—— 1976 *J. Phys. A: Math. Gen.* **9** 1207–14